Joint Availability Importance on Markov Model

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***Abstract*—This paper extends the idea of availability importance from the first-order partial derivative of the system availability to the second-order partial derivative for Markov models. This measure expresses the strength of correlation between two transition rates. We found that a special matrix can be efficiently combined with the procedure of analysis for a continuous-time Markov chain. Numerical experiments demonstrate the effectiveness of our proposal.**

Keywords—reliability, reliability importance, joint reliability importance, joint availability importance

I. INTRODUCTION

How to design a highly reliable system is today's significant issue in the QoS research field. While many topics exist in this field, reliability is significant because research for QoS is to find how to realize high satisfaction for users and reliable service is a key to provide such high satisfaction. Therefore, this paper focuses on reliability issue. Especially, we focus on the topic of reliability importance. This is very useful to find which component should be improved with high priority, under the budget constraint.

The Birnbaum importance [1] was the first measure to be used in reliability importance analysis. It is defined as the partial derivative of the system reliability with respect to the component reliability for a non-repairable system. Birnbaum pointed out that this measure for component i increases as the effect of the reliability of component i on the system reliability grows larger effect to system reliability. By using the Birnbaum importance, designers can easily determine the priority with which components should be improved from the reliability point of view when they face budget limitations.

Availability importance is an extension of the idea from a non-repairable system to a repairable one. It is defined as the partial derivative of the system availability with respect to the component failure rate, component repair rate, or other parameter affecting the system availability. Ref. [2] defined it for combinatorial models, whereas Refs. [3][4] defined it for Markov models. Ref. [5] gave an application.

Here, one question arises. Why do we not have an importance measure based on the second-order partial derivative for a repairable system, even though we do have one based on the second-order partial derivative for a non-repairable system [6][[7]?

Now, we propose an extension of the availability importance from first- to the second-order partial derivative of system availability. We call our measure 'joint availability importance'. We also propose an efficient method to evaluate it for Markov models. Numerical experimental results on computer certify its effectiveness.

II. PREPARATION

We assume that the system consists of n components. The natural number i is the identifier of each component. If we do not need to distinguish between the system and its components, then the word 'item' is used. An item has two states: 'success' and 'failure'.

For a non-repairable system, the reliability of an item is defined as the probability of it being in the success state. If the item is a system, we call it the system reliability, whereas if the item is a component, we call it the component reliability. R denotes the system reliability and R_i denotes the component reliability of component i.

For example, consider the system structure expressed by the reliability block diagram illustrated in Fig. 1. Its system reliability is determined below, where $i = 1, 2, 3$ as in Fig. 1. (We sometimes omit '×' from equations for simplicity.)

$$R = R_1 + R_2R_3 - R_1R_2R_3. \quad (1)$$

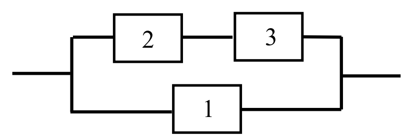

Fig. 1. Reliability block diagram.

The mean time between failures (MTBF), mean time to repair (MTTR), failure rate, repair rate, and availability (A) for a repairable system are defined below.

MTBF: Average time from repair to the next failure of an item

MTTR: Average time from occurrence of a failure to repair of an item

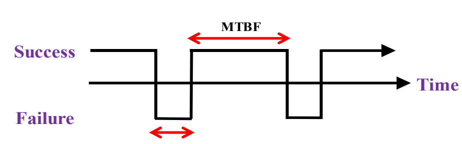

Fig. 2. Success and failure of an item.

Failure rate: $\frac{1}{\mathrm{MTBF}}$ Repair rate: $\frac{1}{\mathrm{MTTR}}$

Availability: $\frac{\mathrm{MTBF}}{\mathrm{MTBF+MTTR}}$

We assume that the system satisfies the conditions of a continuous-time Markov chain (CTMC) [2][3][9] for a repairable system, where the behavior of the system is expressed in terms of a transition diagram. Fig. 3 is an example of a transition diagram.

As mentioned above, states are categorized as success states and failure states. Success states indicate the success of the system, and failure states indicate the failure of the system.

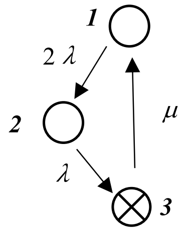

Fig. 3. Example of transition diagram.

In the figures of this paper, a success state is illustrated by an ◯ and a failure state is illustrated by an ⊗. Each state has its own number (state number) from 1, 2, … , *m*. These numbers are written in italics in the figures. $w_{i,j}$ denotes the transition rate from state *i* to state *j*. In Fig. 3, $w_{1,2} = 2\lambda$, $w_{1,3} = 0$, $w_{2,1} = 0$, $w_{2,3} = \lambda$, $w_{3,1} = \mu$, $w_{3,2} = 0$.

Suppose that H_Q (Q = 1, 2, ..) is a two-by-two matrix written as

$$H_Q = \begin{bmatrix} h_{1,1}(Q) & h_{1,2}(Q) \\ h_{2,1}(Q) & h_{2,2}(Q) \end{bmatrix}.$$

We will denote the matrix $\begin{bmatrix} h_{1,1}(1) & h_{1,2}(1) \\ h_{2,1}(1) & h_{2,2}(1) \\ h_{1,1}(2) & h_{1,2}(2) \\ h_{2,1}(2) & h_{2,2}(2) \end{bmatrix}$ by $\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ and the matrix

$$\begin{bmatrix} h_{1,1}(1) & h_{1,2}(1) & h_{1,1}(2) & h_{1,2}(2) \\ h_{2,1}(1) & h_{2,2}(1) & h_{2,1}(2) & h_{2,2}(2) \\ h_{1,1}(3) & h_{1,2}(3) & h_{1,1}(4) & h_{1,2}(4) \\ h_{2,1}(3) & h_{2,2}(3) & h_{2,1}(4) & h_{2,2}(4) \end{bmatrix}$$

by $\begin{bmatrix} H_1 \; H_2 \\ H_3 \; H_4 \end{bmatrix}$. Similar expressions will be used for other matrices. We sometimes use '/' to denote the division operation.

III. Analysis of CTMC

Here, we review the basic techniques of analysis of CTMC that will help the reader to understand our proposal.

An infinitesimal generator matrix *G* is used to evaluate the system availability *A* for CTMC [9]. *G* is an *m* × *m* matrix with the following properties, where *m* is the number of states and *u* and *v* are natural numbers.

Property 1. The (*u*, *v*)-th element of *G* is $w_{u,v}$ if $u \neq v$.
Property 2. The (*u*, *v*)-th element is the negative of the sum of the other elements in row *u*

Property 2 implies that the row sum of *G* is zero.

The following procedure is used to evaluate the system availability with *G*.

Step 1. Let $\pi = (P_1, P_2, \ldots, P_m)$ and $\boldsymbol{0}$ be a row vector consisting of *m* zeros. The following matrix equation is true.

$$\pi G = \boldsymbol{0}$$

Step 2. $\pi G = \boldsymbol{0}$ implies that we have *m* linear equations. We replace one of them with the following equation.

$$P_1 + P_2 + \ldots + P_m = 1$$

We can determine π from these equations because the problem is one of solving *m* linear equations with *m* unknown variables P_k.

Step 3. We obtain the system availability *A* by summing P_k over all success states.

An example solution for the model in Fig. 3 is demonstrated below.

Step 1. $G = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ 0 & -\lambda & \lambda \\ \mu & 0 & -\mu \end{bmatrix}$. From this and $\pi G = \boldsymbol{0}$, we get

$$[P_1, P_2, P_3] \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ 0 & -\lambda & \lambda \\ \mu & 0 & -\mu \end{bmatrix} = [0 \quad 0 \quad 0].$$

Step 2. Accordingly, we obtain the following equations.

$$\begin{aligned} P_1 \times (-2\lambda) & & + P_3\mu &= 0 \\ P_1 \times 2\lambda & - P_2\lambda & &= 0 \\ & P_2\lambda & - P_3\mu &= 0 \end{aligned}$$

After replacing the last equation with $P_1 + P_2 + P_3 = 1$, the above can be expressed in terms of a matrix equation:

$$\begin{bmatrix} -2\lambda & 0 & \mu \\ 2\lambda & -\lambda & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, P_1, P_2 and P_3 can be computed from

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -2\lambda & 0 & \mu \\ 2\lambda & -\lambda & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2\lambda^2 + 3\lambda\mu} \begin{bmatrix} -\lambda & \mu & \lambda\mu \\ -2\lambda & -2\lambda - \mu & 2\lambda\mu \\ 3\lambda & 2\lambda & 2\lambda^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We obtain

$$P_1 = \frac{\lambda\mu}{2\lambda^2+3\lambda\mu} = \frac{\mu}{2\lambda+3\mu}, \; P_2 = \frac{2\lambda\mu}{2\lambda^2+3\lambda\mu} = \frac{2\mu}{2\lambda+3\mu},$$
$$P_3 = \frac{2\lambda^2}{2\lambda^2+3\lambda\mu} = \frac{2\lambda}{2\lambda+3\mu}.$$

Step 3. Availability of the model in Fig. 3 is evaluated as

$$A = P_1 + P_2 = \frac{\mu}{2\lambda+3\mu} + \frac{2\mu}{2\lambda+3\mu} = \frac{3\mu}{2\lambda+3\mu}. \quad (2)$$

If $\lambda = 0.001$ and $m = 2.0$, then $A = 0.99967$ for this example.

IV. Previous Research

A. Reliability Importance for Non-Repairable Systems

As mentioned above, Ref. [1] proposed a measure for reliability importance analysis, called the Birnbaum importance, which we will denote by BI_i. This measure is defined as the partial derivative of system reliability with respect to the component reliability, that is, $BI_i \equiv \frac{\partial R}{\partial R_i}$.

Ref. [1] emphasized that a component with a large Birnbaum importance should be given higher priority for improvement.

For example, BI_1, BI_2 and BI_3 for Fig. 1 are as follows.

$$BI_1 = \frac{\partial R}{\partial R_1} = \frac{\partial(R_1+R_2R_3-R_1R_2R_3)}{\partial R_1} = 1 - R_2R_3$$

$$BI_2 = \frac{\partial R}{\partial R_1} = \frac{\partial(R_1+R_2R_3-R_1R_2R_3)}{\partial R_2} = R_3 - R_1R_3$$

$$BI_3 = \frac{\partial R}{\partial R_1} = \frac{\partial(R_1+R_2R_3-R_1R_2R_3)}{\partial R_2} = R_2 - R_1R_2$$

If $R_1 = R_2 = R_3 = 0.99$, then $BI_1 = 0.0199$, $BI_2 = BI_3 = 0.099$ if $R_1 = R_2 = R_3 = 0.99$. Therefore, improving component 1 should be given high priority.

Ref. [6][7] extended the idea of Birnbaum to the second-order partial derivative, calling the resulting measure the joint reliability importance. We denote it by JRI(u, v), where u and v are identifiers of components. That is,

$$JRI(u, v) \equiv \frac{\partial^2 R}{\partial R_u \partial R_v}.$$

The following are interpretations of the joint reliability importance [6][7].

I1. JRI(u, v) > 0 indicates that one component becomes more important when the other is functioning (synergy);

I2. JRI(u, v) < 0 indicates that one component becomes less important when the other is functioning (diminishing returns);

I3. JRI(u, v) = 0 indicates that one component's importance is unchanged by the function of the other.

For example of Fig. 1,

$$JRI(1, 2) = \frac{\partial^2 R}{\partial R_1 \partial R_2} = \frac{\partial(R_1+R_2R_3-R_1R_2R_3)}{\partial R_1 \partial R_2} = -R_3 < 0.$$

$$JRI(2, 3) = \frac{\partial^2 R}{\partial R_2 \partial R_3} = \frac{\partial(R_1+R_2R_3-R_1R_2R_3)}{\partial R_2 \partial R_3} = 1 - R_1 > 0$$

$$JRI(3, 1) = \frac{\partial^2 R}{\partial R_3 \partial R_1} = \frac{\partial(R_1+R_2R_3-R_1R_2R_3)}{\partial R_3 \partial R_1} = -R_2 < 0$$

These imply that components 1 and 2 and 3 and 1 show diminishing returns, while components 2 and 3 show synergy.

Refs. [6][7] emphasized that such information is very important in reliability design and management of systems, including systems modeled as graphs, as is commonly done when designing communications networks.

B. Reliability Importance for Repairable Systems

Ref. [2] was the first to show an extension of the idea of Birnbaum to repairable systems. Refs. [3][4] devised an extension to Markov models. They defined availability importance as the partial derivative of the system availability A with respect to the transition rate (component failure rate, component repair rate, or other transition rate affecting the system availability) on the Markov model.

For the case shown in Fig. 3, A is obtained from Eq. (2) in Section II. Here,

$$\frac{\partial A}{\partial \lambda} = \frac{-6\mu}{(2\lambda + 3\mu)^2}, \quad \frac{\partial A}{\partial \mu} = \frac{6\lambda}{(2\lambda + 3\mu)^2}.$$

If $\lambda = 0.001$ and $\mu = 2.0$, then $\frac{\partial A}{\partial \lambda} = -0.333$ and $\frac{\partial A}{\partial \mu} = 0.000167$.

These results imply that, for this example, an improvement to reduce the occurrences of failures (reducing λ) is more effective than an improvement to speed up repairs (increasing μ). Refs. [6][7] emphasized that such information is very useful in the reliability design of systems.

C. Problem of Previous Research

While the joint reliability importance defined as the second-order partial derivative of system reliability is known to be useful for non-repairable systems, the utility of the second-order partial derivative of system availability for repairable systems has yet to be investigated.

V. PROPOSAL

A. Joint Availability Importance

Here, we define a new measure, called the 'joint availability importance', as the second-order partial derivative of the system availability of a repairable system with respect to two transitions rates in its state transition diagram. That is, the joint availability importance JAI(α, β) for transition rates α and β is

$$\text{JAI}(\alpha, \beta) \equiv \frac{\partial^2 A}{\partial \alpha \partial \beta}$$

For the case shown in Fig. 3, A is obtained from Eq. (2) in Section II. Here,

$$\text{JAI}(\lambda, \mu) = \frac{\partial^2 A}{\partial \lambda \partial \mu} = \frac{-12\lambda + 18\mu}{(2\lambda + 3\mu)^3}.$$

If $\lambda = 0.0001$ and $\mu = 0.5$, then JAI(λ, μ) = 2.665.

B. Interpretation of Joint Availability Importance

The joint availability importance is more difficult to interpret than the joint reliability importance. We can explain this by using a simple case of a system consisting of a single component with its failure rate and repair rate denoted by λ and μ, respectively.

In this example, the system availability A is expressed as

$$A = \frac{\mu}{\lambda + \mu} \quad (3)$$

From the second partial derivatives, we obtain

$$\frac{\partial^2 A}{\partial \lambda} = \frac{-\mu}{(\lambda + \mu)^2} < 0, \quad \frac{\partial^2 A}{\partial \mu} = \frac{\lambda}{(\lambda + \mu)^2} > 0.$$

The former implies that A is a function of λ showing a monotone decrease, while the latter implies that A is the function of μ showing a monotone increase.

If we assume $\mu > \lambda$, then

$$\text{JAI}(\lambda, \mu) = \frac{\partial^2 A}{\partial \lambda \partial \mu} = \frac{\mu - \lambda}{(\lambda + \mu)^3} > 0$$

and the slope of $A = \frac{\mu}{\lambda + \mu}$ plotted against λ becomes shallower like in Fig. 4 when the repair rate changes from μ_x to μ_y and $\mu_x < \mu_y$.

Thus, in this case, JAI(λ, μ) > 0 indicates that λ becomes less important when μ becomes large (diminishing returns).

On the other hand, if we recognize μ as the input variable, the slope of A against μ becomes steeper like in Fig. 5 when the failure rate changes λ_x to λ_y and $\lambda_x < \lambda_y$ under the same assumption of $\mu > \lambda$.

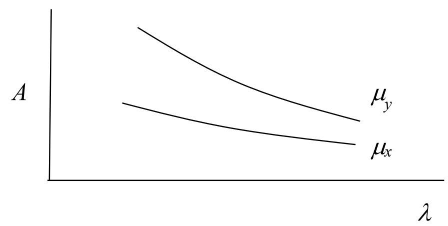

Fig. 4. Plots of $A = \frac{\mu}{\lambda+\mu}$ versus λ.

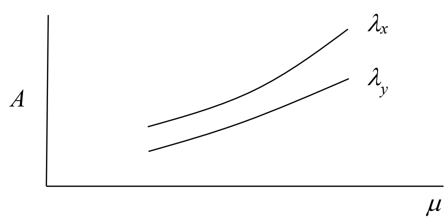

Fig. 5. Plots of $A = \frac{\mu}{\lambda+\mu}$ versus μ.

Thus, in this case, JAI(λ, μ) > 0 indicates that μ becomes more important when λ becomes large (synergy) .

As shown above, JAI(λ, μ) > 0 implies sometimes diminishing returns and sometimes synergy. This is quite different from the joint reliability importance explained in Subsection A of Section IV. Therefore, we will limit the interpretation of the joint availability importance to a simpler one as described below, because the detailed interpretation of f the joint availability importance taking a positive or negative value seems to be difficult for Markov models at present.

I1'. If $JAI(\alpha, \beta) = 0$, there is no correlation between α and β.
I2'. As $|JAI(\alpha, \beta)|$ becomes larger, the strength of the correlation between α and β becomes larger.

C. Evaluation Method

Readers might think that it is difficult to compute the partial derivative when the expression of the pre-derivative function is complicated, because of the following logic.

1. If $f = f_1 \times f_2$ then $\frac{\partial f}{\partial x} = \frac{\partial f_1}{\partial x} \times f_2 + f_1 \times \frac{\partial f_2}{\partial x}$
2. That is, if we have a single multiplication in the expression of f then the number of multiplications becomes two after executing partial derivative to f.
3. Therefore, if we have m multiplications in the expression of f then the number of multiplications is estimated to be2^m after taking the partial derivative.
4. Accordingly, the computation time of the partial derivative of f increases exponentially relative to the number of multiplications in the expression of f.

The above logic further suggests it would be even more difficult to compute the second-order partial derivative of a complicated function f. That's why evaluating the joint availability importance has been thought to be prohibitively complicated when the number of states of the Markov model is large.

However, recent information technologies have given us easy techniques with which to compute partial derivatives. The matrix approach is one of them [10][11].

Here, i we define matrix $\varphi_x(f)$ for f as

$$\varphi_x(f) \equiv \begin{bmatrix} f & 0 \\ \frac{\partial f}{\partial x} & f \end{bmatrix},$$

then the following equations are true for f_1 and f_2, where $\varphi_x(f_1) / \varphi_x(f_2) \equiv \varphi_x(f_1) \times \varphi_x(f_2)^{-1}$. ($\varphi_x(f_2)^{-1}$ is the matrix inverse of $\varphi_x(f_2)$.)

$$\varphi_x(f_1 + f_2) = \varphi_x(f_1) + \varphi_x(f_2) \tag{4}$$
$$\varphi_x(f_1 - f_2) = \varphi_x(f_1) - \varphi_x(f_2) \tag{5}$$
$$\varphi_x(f_1 \times f_2) = \varphi_x(f_1) \times \varphi_x(f_2) \tag{6}$$
$$\varphi_x(f_1 / f_2) = \varphi_x(f_1) / \varphi_x(f_2) \tag{7}$$

For example, if $f = \frac{3\mu}{2\lambda+3\mu}$, by using Eqs. (4)-(7), we find that

$$\varphi_\lambda(f) = \begin{bmatrix} f & 0 \\ \frac{\partial f}{\partial \lambda} & f \end{bmatrix} = 3\begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}\left(2\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} + 3\begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} \frac{3\mu}{2\lambda+3\mu} & 0 \\ \frac{-6\mu}{(2\lambda+3\mu)^2} & \frac{3\mu}{2\lambda+3\mu} \end{bmatrix}$$

and

$$\frac{\partial f}{\partial \lambda} = \frac{-6\mu}{(2\lambda+3\mu)^2}.$$

Ref. [10] showed that, when we use this matrix approach, the computational complexity of obtaining the partial derivative of f is proportional to that of obtaining the output of f.

Ref. [11] extended it to the second-order partial derivative case. That is, if we define

$$\varphi_{x,y}(f) \equiv \begin{bmatrix} f & 0 & 0 & 0 \\ \frac{\partial f}{\partial y} & f & 0 & 0 \\ \frac{\partial f}{\partial x} & 0 & f & 0 \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & f \end{bmatrix},$$

then the following equations are true for functions f_1 and f_2.

$$\varphi_{x,y}(f_1 + f_2) = \varphi_{x,y}(f_1) + \varphi_{x,y}(f_2)$$
$$\varphi_{x,y}(f_1 - f_2) = \varphi_{x,y}(f_1) - \varphi_{x,y}(f_2)$$
$$\varphi_{x,y}(f_1 \times f_2) = \varphi_{x,y}(f_1) \times \varphi_{x,y}(f_2)$$

$$\varphi_{x,y}(f_1 / f_2) = \varphi_{x,y}(f_1) / \varphi_{x,y}(f_2)$$

Therefore, if $f = \frac{3\mu}{2\lambda+3\mu}$, we obtain

$$\varphi_{\lambda,\mu}(f) = \begin{bmatrix} f & 0 & 0 & 0 \\ \frac{\partial f}{\partial \mu} & f & 0 & 0 \\ \frac{\partial f}{\partial \lambda} & 0 & f & 0 \\ \frac{\partial^2 f}{\partial\lambda\partial\mu} & \frac{\partial f}{\partial \lambda} & \frac{\partial f}{\partial \mu} & f \end{bmatrix}$$

$$= 3\begin{bmatrix} \mu & 0 & 0 & 0 \\ 1 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 1 & \mu \end{bmatrix} \times \left(2\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & \lambda \end{bmatrix} + 3\begin{bmatrix} \mu & 0 & 0 & 0 \\ 1 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 1 & \mu \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} \frac{3\mu}{2\lambda+3\mu} & 0 & 0 & 0 \\ \frac{6\lambda}{(2\lambda+3\mu)^2} & \frac{3\mu}{2\lambda+3\mu} & 0 & 0 \\ \frac{-6\mu}{(2\lambda+3\mu)^2} & 0 & \frac{3\mu}{2\lambda+3\mu} & 0 \\ \frac{-12\lambda+18\mu}{(2\lambda+3\mu)^3} & \frac{-6\mu}{(2\lambda+3\mu)^2} & \frac{6\lambda}{(2\lambda+3\mu)^2} & \frac{3\mu}{2\lambda+3\mu} \end{bmatrix}$$

Thus, $\frac{\partial^2 f}{\partial\lambda\partial\mu} = \frac{-12\lambda+18\mu}{2\lambda+3\mu}$

Ref. [11] showed that the complexity of computing the second partial derivative of f is proportional to the computational complexity of obtaining the output of f.

Now we can apply the matrix approach of ref. [11] and obtain JAI(α, β). This entails executing Step 2 of Section III by replacing every element z of the matrix and vectors with $\varphi_{\alpha,\beta}(z)$.

The new steps can be easily executed on software specific to matrix computations, such as MATLAB.

D. *Motivational Example*

Let us demonstrate an example of applying the matrix approach of ref. [11] to the steps of Section III for the case of Fig. 3, where we define $\varphi_{\alpha,\beta}(M)$ to be the matrix obtained by replacing any element of z of matrix M with $\varphi_{\alpha,\beta}(z)$.

Step 1. $G = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ 0 & -\lambda & \lambda \\ \mu & 0 & -\mu \end{bmatrix}$. From this and $\pi G = 0$, we get

$$[P_1, P_2, P_3]\begin{bmatrix} -2\lambda & 2\lambda & 0 \\ 0 & -\lambda & \lambda \\ \mu & 0 & -\mu \end{bmatrix} = [0 \quad 0 \quad 0]$$

Step 2. From Step 1, we obtain the following equations.

$$\begin{aligned} P_1 \times (-2\lambda) \qquad\qquad + P_3\mu &= 0 \\ P_1 \times 2\lambda - P_2\lambda \qquad\qquad &= 0 \\ P_2\lambda - P_3\mu &= 0 \end{aligned}$$

The above can be expressed in terms of a matrix equation:

$$\begin{bmatrix} -2\lambda & 0 & \mu \\ 2\lambda & -\lambda & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By applying $\varphi_{\lambda,\mu}(\ )$ to every element of the matrix (vector is a kind of matrix) in this equation, we obtain

$$\begin{bmatrix} -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mu & 0 & 0 \\ -2 & 0 & -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & -2 & 0 & -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mu \\ 2\lambda & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\lambda & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2\lambda & 0 & -1 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2\lambda & 0 & -1 & 0 & -\lambda & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} P_1 & 0 & 0 & 0 \\ \frac{\partial P_1}{\partial\mu} & P_1 & 0 & 0 \\ \frac{\partial P_1}{\partial\lambda} & 0 & P_1 & 0 \\ \frac{\partial^2 P_1}{\partial\lambda\partial\mu} & \frac{\partial P_1}{\partial\lambda} & \frac{\partial P_1}{\partial\mu} & P_1 \\ P_2 & 0 & 0 & 0 \\ \frac{\partial P_2}{\partial\mu} & P_2 & 0 & 0 \\ \frac{\partial P_2}{\partial\lambda} & 0 & P_2 & 0 \\ \frac{\partial^2 P_2}{\partial\lambda\partial\mu} & \frac{\partial P_2}{\partial\lambda} & \frac{\partial P_2}{\partial\mu} & P_2 \\ P_3 & 0 & 0 & 0 \\ \frac{\partial P_3}{\partial\mu} & P_3 & 0 & 0 \\ \frac{\partial P_3}{\partial\lambda} & 0 & P_3 & 0 \\ \frac{\partial^2 P_2}{\partial\lambda\partial\mu} & \frac{\partial P_3}{\partial\lambda} & \frac{\partial P_3}{\partial\mu} & P_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Accordingly,

$$\begin{bmatrix} P_1 & 0 & 0 & 0 \\ \frac{\partial P_1}{\partial\mu} & P_1 & 0 & 0 \\ \frac{\partial P_1}{\partial\lambda} & 0 & P_1 & 0 \\ \frac{\partial^2 P_1}{\partial\lambda\partial\mu} & \frac{\partial P_1}{\partial\lambda} & \frac{\partial P_1}{\partial\mu} & P_1 \\ P_2 & 0 & 0 & 0 \\ \frac{\partial P_2}{\partial\mu} & P_2 & 0 & 0 \\ \frac{\partial P_2}{\partial\lambda} & 0 & P_2 & 0 \\ \frac{\partial^2 P_2}{\partial\lambda\partial\mu} & \frac{\partial P_2}{\partial\lambda} & \frac{\partial P_2}{\partial\mu} & P_2 \\ P_3 & 0 & 0 & 0 \\ \frac{\partial P_3}{\partial\mu} & P_3 & 0 & 0 \\ \frac{\partial P_3}{\partial\lambda} & 0 & P_3 & 0 \\ \frac{\partial^2 P_2}{\partial\lambda\partial\mu} & \frac{\partial P_3}{\partial\lambda} & \frac{\partial P_3}{\partial\mu} & P_3 \end{bmatrix} = \begin{bmatrix} -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mu & 0 & 0 \\ -2 & 0 & -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & -2 & 0 & -2\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mu \\ 2\lambda & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\lambda & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2\lambda & 0 & -1 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2\lambda & 0 & -1 & 0 & -\lambda & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\mu}{2\lambda+3\mu} & 0 & 0 & 0 \\ \frac{2\lambda}{(2\lambda+3\mu)^2} & \frac{\mu}{2\lambda+3\mu} & 0 & 0 \\ \frac{-2\mu}{(2\lambda+3\mu)^2} & 0 & \frac{\mu}{2\lambda+3\mu} & 0 \\ \frac{-4\lambda+6\mu}{(2\lambda+3\mu)^3} & \frac{-2\mu}{(2\lambda+3\mu)^2} & \frac{2\lambda}{(2\lambda+3\mu)^2} & \frac{\mu}{2\lambda+3\mu} \\ \frac{2\mu}{2\lambda+3\mu} & 0 & 0 & 0 \\ \frac{4\lambda}{(2\lambda+3\mu)^2} & \frac{2\mu}{2\lambda+3\mu} & 0 & 0 \\ \frac{-4\mu}{(2\lambda+3\mu)^2} & 0 & \frac{2\mu}{2\lambda+3\mu} & 0 \\ \frac{-8\lambda+12\mu}{(2\lambda+3\mu)^3} & \frac{-4\mu}{(2\lambda+3\mu)^2} & \frac{4\lambda}{(2\lambda+3\mu)^2} & \frac{2\mu}{2\lambda+3\mu} \\ \frac{2\lambda}{2\lambda+3\mu} & 0 & 0 & 0 \\ \frac{-6\lambda}{(2\lambda+3\mu)^2} & \frac{2\lambda}{2\lambda+3\mu} & 0 & 0 \\ \frac{6\mu}{(2\lambda+3\mu)^2} & 0 & \frac{2\lambda}{2\lambda+3\mu} & 0 \\ \frac{12\lambda-18\mu}{(2\lambda+3\mu)^3} & \frac{6\mu}{(2\lambda+3\mu)^2} & \frac{-6\lambda}{(2\lambda+3\mu)^2} & \frac{2\lambda}{2\lambda+3\mu} \end{bmatrix}.$$

Finally, we obtain

$$\frac{\partial^2 P_1}{\partial\lambda\partial\mu} = \frac{-4\lambda+6\mu}{(2\lambda+3\mu)^3}, \frac{\partial^2 P_2}{\partial\lambda\partial\mu} = \frac{-8\lambda+12\mu}{(2\lambda+3\mu)^3}, \frac{\partial^2 P_2}{\partial\lambda\partial\mu} = \frac{12\lambda-18\mu}{(2\lambda+3\mu)^3}$$

Step 3. JRI(λ, μ) is evaluated as

$$\text{JRI}(\lambda,\mu) = \frac{\partial^2 A}{\partial\lambda\partial\mu} = \frac{\partial^2 P_1}{\partial\lambda\partial\mu} + \frac{\partial^2 P_2}{\partial\lambda\partial\mu}$$

$$= \frac{-4\lambda+6\mu}{(2\lambda+3\mu)^3} + \frac{-8\lambda+12\mu}{(2\lambda+3\mu)^3} = \frac{-12\lambda+18\mu}{(2\lambda+3\mu)^3}.$$

If $\lambda = 0.001$ and $\mu = 2.0$, then JRI(λ, μ) $= 0.16645 > 0$ for this example.

VI. NUMERICAL EXPERIMENTS

The experiment was executed in the following environment.

OS: Windows 10 home
CPU: Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz 1.99 GHz
RAM: 8.00GB, Language: MATLAB R2019b

The target system is illustrated in Fig. 6. Ref. [3] evaluates the availability importance for this model.

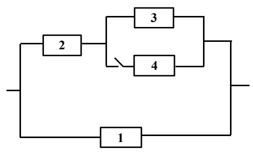

Fig. 6. Target model.

The failure rate of component a is expressed as λ_a and the repair rate is expressed as μ_a for a = 1, 2, 3, and 4. Component 4 is a cold spare which not used unless component 3 fails. It is also assumed that the failure rate of component 1 increases from λ_1 to $\overline{\lambda}_1$ when the load on component 1 increases due to the failure of any of the components 2, 3 and 4. The values of these parameters are given below.

$\lambda_1 = 0.01, \lambda_2 = 0.01, \lambda_3 = 0.01, \lambda_4 = 0.01, \overline{\lambda}_1 = 0.015,$
$\mu_1 = 0.05, \mu_2 = 0.05, \mu_3 = 0.05, \mu_4 = 0.05$

The state transition diagram is illustrated in Fig. 7.

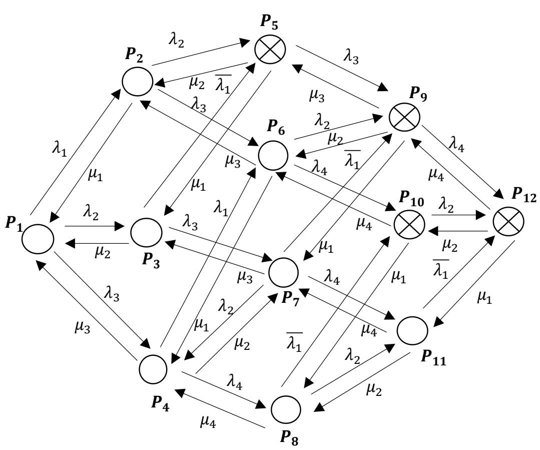

Fig. 7. State transition diagram.

For this model, we obtain JAI(λ_1, μ_3) = −1.61 and JAI(λ_1, μ_4) = −0.415. These imply that the strength of correlations of the component failure rate and repair rate between components 1and 3 and the component failure rate and repair rate between 1 and 4 are quite different, whose reason is the cold spare of component 4. It would be difficult to find such a difference without evaluating the joint availability importance.

The computation time needed to evaluate both JAI(λ_1, μ_3) and JAI(λ_1, μ_4) is less than 0.1 second on a computer.

V. CONCLUSION

We proposed to extend the idea of availability importance from the first-order partial derivative of the system availability to the second-order partial derivative for Markov models. This measure expresses the strength of correlation between two transition rates of a Markov model. We found that a special matrix can be efficiently combined with the procedure to execute an analysis of CTMC to obtain the joint availability importance.

Numerical experimental results showed the effectiveness of our proposal.

Future work will include:

1. a more detailed study of the interpretation of joint availability importance
2. application of more practical models in order to derive useful findings in reliability design.
3. development of importance measures based on much higher-order partial derivatives.

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